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A unified study of inversion of an integral equation with the I - function of two variables as its kernel

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ABSTRACT

The object of this paper is to solve an integral equation of convolution from having the I -function of two variables as its kernel. Some special cases are also given in the end.

Keywords: Laplace Transform; Lerch's Theorem; I -function. (2000 Mathematics Subject Classification: 33C99)

1. INTRODUCTION

1.1. Definition and results

The Laplace Transform

$$F(p)=\int_0^\infty e^{-pt} f(t)dt,\ \mathrm{Re}(p)>0 \tag{1}$$

Is represented by $F(p)=L\{f(t)\}$.

Erdelyi (1954),

If $F(p)=L\{f(t)\}$ then

$$e^{-at} f(t)=F(p+a). \tag{2}$$

If $L\{f(t)\}=F(p), f(0)=f^{'}(0)=...=f^{n-1}(0)=0$ and $f^{n}(t)$ is continuous

then

$$L\{f^{n}(t)\}=p^{n}F(p). \tag{3}$$

If $L\{f_1(t)\}=F_1(p)$ and $L\{f_2(t)\}=F_2(p)$, then

$$\int_0^t f_1(u) f_2(t-u) du = F_1(p) F_2(p) \qquad (4)$$

The I -function introduced by Saxena (1982) will be represented and defined as follows:

$$I[Z] = I^{m,n}_{p_i,q_i:r}[Z] = I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{smallmatrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{smallmatrix}\right.\right] = \frac{1}{2\pi\omega}\int_L \chi(\xi) d\xi \qquad (5)$$

where $\omega = \sqrt{-1}$

$$\chi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji})\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji})\right\}} \qquad (6)$$

$p_{i,}q_i(i=1,...,r),m,n$ are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i,(i=1,...,r),r$ is finite $\alpha_j,\beta_j,\alpha_{ij},\beta_{ji}$ are real and $a_{j,}b_j,a_{ji},b_{ji}$ are complex numbers such that

$\alpha_j(b_h+v) \ne \beta_h(a_j - v - k)$ for $v,k = 01,2,...$

We shall use the following notations:

$A^* = (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}; B^* = (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}$ The following results are used in the sequel:

$$t^h I^{1,1}_{2,1}\left[zt^{-k}\left|\begin{smallmatrix}(1-v,1),(1+h,k)\\(0,1)\end{smallmatrix}\right.\right] = p^{-1-h}(1+zp^k)^{-v}\Gamma(v) \qquad (7)$$

Provided that $\mathrm{Re}(p)>0, 2>k>0, \mathrm{Re}(1+h+kv)>0, |\arg zp^k| < \frac{\pi}{2}(2-k)$

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} I^{m,n}_{p_i,q_i:r}\left[z_1 x^{\lambda}\left|\begin{smallmatrix}A^*\\B^*\end{smallmatrix}\right.\right] I^{g,h}_{u_i,v_i:r}\left[z_2(1-x)^{\mu}\left|\begin{smallmatrix}A^{**}\\B^{**}\end{smallmatrix}\right.\right] dx =$$

$$I^{0,0:m,n+1:\ \ g,h+1}_{0,1:p_i+1,q_i+1:r:u_i+1,v_i+1:r}\left[\begin{smallmatrix}z_1\\z_2\end{smallmatrix}\left|\begin{smallmatrix}(1-\alpha,\lambda),A^{**},A^{***}\\(1-\alpha-\beta,\lambda,\mu),B^{**},B^{***}\end{smallmatrix}\right.\right] \qquad (8)$$

Where

$$\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \lambda,\mu>0, \mathrm{Re}\left(\alpha+\lambda\frac{d_j}{D_j}\right)>0,$$

$$\mathrm{Re}\left(\beta+\lambda\frac{f_j}{F_j}\right)>0, (j=1,2,...m; k=1,2,...,g)$$

$|\arg z_1| < \frac{1}{2}\pi\Delta_1, |\arg z_2| < \frac{1}{2}\pi\Delta_2; \Delta_1,\Delta_2 > 0$ where

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$$\Delta_1 = \sum_{j=1}^{m} D_j - \sum_{j=m+1}^{q_i} D_{ji} + \sum_{j=1}^{n} C_j - \sum_{j=n+1}^{p_i} C_{ji}$$

And

$$\Delta_2 = \sum_{j=1}^{g} F_j - \sum_{j=g+1}^{v_i} F_{ji} + \sum_{j=1}^{h} E_j - \sum_{j=h+1}^{u_i} E_{ji}$$

$$A^{**} = (c_j, C_j)_{1,n}, (c_{ji}, C_{ji})_{n+1,p_i}; B^{**} = (d_j, D_j)_{1,m}, (d_{ji}, D_{ji})_{m+1,q_i}$$

$$A^{***} = (e_j, E_j)_{1,h}, (e_{ji}, E_{ji})_{h+1,u_i}; B^{***} = (f_j, F_j)_{1,g}, (f_{ji}, F_{ji})_{g+1,v_i}$$

The I -function of two variables introduced by Prasad (1986) will be represented and defined as follows:

$$I[z_1, z_2] = I^{0,n_2:(m'.n'):(m",n")}_{p_2,q_2:(p',q'):(p",q")}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left|\begin{matrix} (a_{2j}:\alpha'_{2j},\alpha"_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p"} \\ (b_{2j}:\beta'_{2j},\beta"_{2j})_{1,q_2}:(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q"} \end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi w)^2}\int_{L_1}\int_{L_2} \phi_1(s_1)\phi_2(s_2)\psi(s_1,s_2) z_1^{s_1} z_2^{s_2} ds_1 ds_2$$

$$e^{-bt} t^h I^{1,1}_{2,1}\left[zt^{-k}\left|\begin{matrix}(1-v,1),(1+h,k)\\(0,1)\end{matrix}\right.\right] = (p+b)^{-1-h}(1+z(p+b)^k)^{-v}\Gamma(v)$$

(9) Where $w = \sqrt{-1}$,

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m^{(i)}} \Gamma\left(b_j^{(i)} - \beta_j^{(i)} s_i\right) \prod_{j=1}^{n^{(i)}} \Gamma\left(1 - a_j^{(i)} + \alpha_j^{(i)} s_i\right)}{\prod_{j=m^{(i)}+1}^{q^{(i)}} \Gamma\left(1 - b_j^{(i)} + \beta_j^{(i)} s_i\right) \prod_{j=n^{(i)}1}^{p^{(i)}} \Gamma\left(a_j^{(i)} - \alpha_j^{(i)} s_i\right)} \quad \forall i \in \{1,2\}$$

(10)

$$= \sum_{r=0}^{\infty} \frac{b^r}{r!} e^{-(b+a)t} t^{r+h_1+h_2+1} I^{0,0:2,1:1,1}_{1,0:2,1:1,1}\left[\begin{matrix} z_1 t^{-k_1} \\ z_2 t^{-k_2} \end{matrix}\left|\begin{matrix}(r+h_1+h_2+2,k_1,k_2),(1-v_1,k_1),(1+h_1,k_1),(1-v_2,1)\\(\ldots:(r+h_1+1,k_1),(0,1),(0,1)\end{matrix}\right.\right]$$

$$\psi(s_1,s_2) = \frac{\prod_{j=1}^{n_2} \Gamma\left(1 - a_{2j} + \sum_{i=1}^{2} a_{2j}^{(i)} s_i\right)}{\prod_{j=n_2+1}^{p_2} \Gamma\left(a_{2j} - \sum_{i=1}^{2} a_{2j}^{(i)} s_i\right) \prod_{j=1}^{q_2} \Gamma\left(1 - b_{2j} + \sum_{i=1}^{2} \beta_{2j}^{i} s_i\right)} \quad (11)$$

2. MAIN RESULT

Theorem: Each of the integral equation

$$g(t) = A\sum_{r=0}^{\infty}\frac{b^r}{r!}\int_0^t \left[(D+a)^{m_1}(D+a+b)^{m_2} f(u)\right] e^{-(a+b)(t-u)}(t-u)^{r+h_1+h_2+1}$$

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$$\times I_{1,0:2,2:1,1}^{0,0:2,1:1,1}\left[\begin{matrix} z_1(t-u)^{-k_1} \\ z_2(t-u)^{-k_2} \end{matrix}\middle| \begin{matrix} (r+h_1+h_2+2,k_1,k_2):(1-v_1,1),(1+h_1,k_1):(1-v_2,1) \\ ...:(r+h_1+1,k_1),(0,1),(0,1) \end{matrix}\right]du \quad (12)$$

And

$$f(t)=B\sum_{r=0}^{\infty}\frac{b^r}{r!}\int_0^t\left[(D+a)^{n_1}(D+a+b)^{n_2}g(u)\right]e^{-(a+b)(t-u)}(t-u)^{r+h'_1+h'_2+1}$$

$$\times I_{1,0:2,2:1,1}^{0,0:2,1:1,1}\left[\begin{matrix} z_1(t-u)^{-k_1} \\ z_2(t-u)^{-k_2} \end{matrix}\middle| \begin{matrix} (r+h'_1+h'_2+2,k_1,k_2):(1-v_1,1),(1+h'_1,k_1):(1-v_2,1) \\ ...:(r+h'_1+1,k_1),(0,1),(0,1) \end{matrix}\right]du \quad (13)$$

In the solution of the other, provided

$$m_1+n_1=h_1+h'_1+2, m_2+n_2=h_2+h'_2+2$$

$$AB\Gamma(v_1)\Gamma(v_2)\Gamma(-v_1)\Gamma(-v_2)=1, 2>k_1>k_2>0$$

$f(0)=f'(0)=...=f^{m_1-1}(0)=0,\ f^{m_1}(t)$ is continuous

$f(0)=f'(0)=...=f^{m_2-1}(0)=0,\ f^{m_2}(t)$ is continuous

$g(0)=g'(0)=...=g^{n_1-1}(0)=0, g^{n_1}(t)$ is continuous

$g(0)=g'(0)=...=g^{n_2-1}(0)=0, g^{n_2}(t)$ is continuous

m_1, m_2, n_1 and n_2 are integers.

D represents differentiation with respect to u and

$$\frac{1}{D+a}f(u)=e^{-au}\int_0^u f(u)e^{au}du$$

2.1. Proof

Let $L\{f(t)\}=F(p)$ and $L\{g(t)\}=G(p)$

Using (2) and (7) (14)

Using (4) in (7)and (1)

$$\Gamma(v_1)p^{-1-h_1}\left[1+z_1p^{k_1}\right]^{-v_1}\Gamma(v_2)(p+b)^{-1-h_2}\left[1+z_2(p+b)^{k_2}\right]^{-v_2}$$

$$=\int_0^t u^{h_1}I_{2,1}^{1,1}\left[z_1u^{-k_1}\middle|\begin{matrix}(1-v_1,1),(1+h_1,k_1)\\(0,1)\end{matrix}\right]e^{-b(t-u)}(t-u)^{h_2}$$

$$\times I_{2,1}^{1,1}\left[z_2(t-u)^{-k_2}\middle|\begin{matrix}(1-v_2,1),(1+h_2,k_2)\\(0,1)\end{matrix}\right]du \quad (15)$$

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$$\text{R.H.S.} = e^{-bt}\int_0^t u^{h_1} I_{2,1}^{1,1}\left[z_1 u^{-k_1} \left|{}^{(1-v_1,1),(1+h_1,k_1)}_{(0,1)}\right.\right] e^{-bu}(t-u)^{h_2}$$

$$\times I_{2,1}^{1,1}\left[z_2 (t-u)^{-k_2} \left|{}^{(1-v_2,1),(1+h_2,k_2)}_{(0,1)}\right.\right] du \qquad (16)$$

Now expand e^{bu} and put $u = tv$ to get (16) in the form:

$$\text{R.H.S.}= = \sum_{r=0}^{\infty}\frac{b^r}{r!} e^{-bt} t^{r+h_1+h_2+1}\int_0^1 v^{h_1+r}(1-v)^{h_2} I_{2,1}^{1,1}\left[z_1 u^{-k_1} v^{-k_1} \left|{}^{(1-v_1,1),(1+h_1,k_1)}_{(0,1)}\right.\right]$$

$$\times I_{2,1}^{1,1}\left[z_2 t^{-k_2} (1-u)^{-k_2} \left|{}^{(1-v_2,1),(1+h_2,k_2)}_{(0,1)}\right.\right] dv \qquad (17)$$

Now evaluating (17) using (8) to get R.H.S.

$$= \sum_{r=0}^{\infty}\frac{b^r}{r!} e^{-bt} t^{r+h_1+h_2+1} I_{1,0:2,1:1,1}^{0,0:2,1:1,1}\left[{}^{z_1 t^{-k_1}}_{z_2 t^{-k_2}} \left|{}^{(r+h_1+h_2+2,k_1,k_2),(1-v_1,k_1),(1+h_1,k_1),(1-v_2,1)}_{(\ldots:(r+h_1+1,k_1),(0,1),(0,1)}\right.\right] \qquad (18)$$

Using (1) and (18), we get

$$\Gamma(v_1)(p+a)^{-1-h_1}\left[1+z_1(p+a)p^{k_1}\right]^{-v_1}\Gamma(v_2)(p+a+b)^{-1-h_2}\left[1+z_2(p+a+b)^{k_2}\right]^{-v_2} \qquad (19)$$

Using (4) and (19) the integral equations (12) and (13) become

$$\begin{aligned} G(p) = A\Gamma(v_1)(p+a)^{m_1-1-h_1}(p+a+b)^{m_2-1-h_2}F(p)\left[1+z_1(p+a)p^{k_1}\right]^{-v_1} \\ \Gamma(v_2)\left[1+z_2(p+a+b)^{k_2}\right]^{-v_2} \end{aligned} \qquad (20)$$

$$\begin{aligned} F(p) = B\Gamma(-v_1)(p+a)^{n_1-1-h_1}(p+a+b)^{n_2-1-h_2}G(p)\left[1+z_1(p+a)p^{k_1}\right]^{-v_1} \\ \Gamma(-v_2)\left[1+z_2(p+a+b)^{k_2}\right]^{-v_2} \end{aligned} \qquad (21)$$

The equations (20) and (21) can be obtained from each other when

$$AB\Gamma(v_1)\Gamma(v_2)\Gamma(-v_1)\Gamma(-v_2) = 1,$$

$m_1 + n_1 = h_1 + h'_1 + 2$ and $m_2 + n_2 = h_2 + h'_2 + 2$

Hence by Lerch's theorem (1962), it follows that each of the integral equations (12) and (13) is the solution of the other.

3. SPECIAL CASES

In the theorem put $k_2 = 1, k_1 = k, v_1 = v, z_1 = z$ and make $z_2 \to 0$ to get the following result involving I -function of one variable.

Each of the integral equations

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$$g(t)=A\sum_{r=0}^{\infty}\frac{b^r}{r!}\int_0^t\left[(D+a)^{m_1}(D+a+b)^{m_2}f(u)\right]e^{-(a+b)(t-u)}(t-u)^{r+h_1+h_2+1}$$

$$\times I_{3,2}^{2,1}\left[z(t-u)^{-k}\left|{}^{(r+h_1+h_2+2,k):(1-v,1),(1+h_1,k)}_{...:(r+h_1+1,k),(0,1)}\right.\right]du \quad (22)$$

And

$$f(t)=B\sum_{r=0}^{\infty}\frac{b^r}{r!}\int_0^t\left[(D+a)^{n_1}(D+a+b)^{n_2}g(u)\right]e^{-(a+b)(t-u)}(t-u)^{r+h'_1+h'_2+1}$$

$$\times I_{3,2}^{2,1}\left[z(t-u)^{-k}\left|{}^{(r+h'_1+h'_2+2,k):(1+v,1),(1+h'_1,k)}_{...:(r+h'_1+1,k),(0,1)}\right.\right]du \quad (23)$$

In the solution of the other, provided the conditions of Theorem are satisfied with

$AB\Gamma(v)\Gamma(-v)=1,$ and $2>k>0$

When $h_1=\alpha, h'_1=\beta, h_2=h'_2=-1, m_1=m, n_1=n, m_2=n_2=0$ and $b\to 0$, (22) and (23) reduces to:

Each of the integral equations

$$g(t)=A\int_0^t\left[(D+a)^m f(u)\right]e^{-(a)(t-u)}(t-u)^{\alpha}$$

$$\times I_{2,1}^{1,1}\left[z(t-u)^{-k}\left|{}^{(1+\alpha,k):(1-v,1)}_{(0,1)}\right.\right]du \quad (24)$$

And

$$f(t)=B\int_0^t\left[(D+a)^n g(u)\right]e^{-(a)(t-u)}(t-u)^{\beta}$$

$$\times I_{2,1}^{1,1}\left[z(t-u)^{-k}\left|{}^{(1+\beta,k):(1+v,1)}_{(0,1)}\right.\right]du \quad (25)$$

is the solution of the other, provided

m and n are integers $m+n=2+\alpha+\beta$

$f(0)=f'(0)=...=f^{m-1}(0)=0, and\ f^m(t)$ is continuous when $m>0$

$g(0)=g'(0)=...=g^{n-1}(0)=0, and\ g^n(t)$ is continuous when $n>0$

$AB\Gamma(v)\Gamma(-v)=1, \quad \mathrm{Re}(1+\alpha+kv)>0, 2>k>0, \mathrm{Re}(1+\beta-kv)>0$

$$D=\frac{d}{du},\frac{1}{D}=\int_0^u du$$

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$$\frac{1}{D+a}f(u)=e^{-au}\int_0^u f(u)e^{au}du$$

(24) and (25) agree with the result given by Nair (1986).

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Conflicts of interests
The authors declare that there are no conflicts of interests.

Data and materials availability
All data associated with this study are present in the paper.

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